RECOLLECTING BASIC THEOREMS OF THE KKM THEORY (Nonlinear Analysis and Convex Analysis)

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ABSTRACT. In our earlier foundational works on the KKM theory, we were based on several KKM type theorems or the Fan-Browder type coincidence theorems. Recently, we obtained three general KKM type theorems A, B, and C for abstract convex spaces. In this paper, we obtain a new coincidence theorem (Theorem D) and recollect that several particular forms of Theorems A-D were applied to establish our earlier foundational works for each of convex spaces, H-spaces, G-convex spaces, and abstract convex spaces.

1. Introduction

The KKM theory, first called by the author [1], is the study on applications of equivalent formulations of the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz in 1929. The KKM theorem provides the foundations for many of the modern essential results in diverse areas of mathematical sciences.

Some of the basic theorems which are useful to applications of the KKM theory were first obtained by Ky Fan, Browder, Granas, and others for convex subsets of topological vector spaces. Later extensions of the theory were due to Lassonde for convex spaces, Horvath for H-spaces, Park for G-convex spaces, and others; see [6,11] and the references therein.

Recently, the KKM theory is extended to abstract convex spaces by the author and we obtained new results in such frame; see [8-13] and the references therein. Moreover, in such frame, we obtained three basic KKM theorems A, B, and C in our works [13-15,17]. Recall that there are large numbers of equivalent formulations, generalizations, and applications of the KKM theorem.

Until now, we have published several papers on the *elements or foundations* of the KKM theory; namely, for convex spaces [1,2], H-spaces [3], generalized convex spaces [4,5,7], and abstract convex spaces [8,9,11,12]. Each of these papers is based on KKM type theorems or Fan-Browder type

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coincidence theorems and concerned with useful fundamental results in the KKM theory.

In the present paper, we obtain a Fan-Browder type coincidence theorem (Theorem D) and show that the basic theorems in [1-5,7-9,11,12] follow from one of Theorems A, B, C, and D.

Section 2 devotes to give some necessary terminology on abstract convex spaces. In Section 3, we introduce Theorems A, B, and C. Section 4 is to deduce a new Fan-Browder type coincidence theorem (Theorem D) from Theorem C. Finally, in Section 5, we recollect several particular forms of Theorems A-D, which were applied to establish our earlier foundational works for each of convex spaces, H-spaces, G-convex spaces, and abstract convex spaces.

2. Abstract convex spaces

For the concepts of abstract convex spaces and KKM spaces, the reader may consult with the references in [8-12].

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$, where $\langle D \rangle$ is the set of all nonempty finite subsets of D.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp., a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E, D, Z)$ [resp., $F \in \mathfrak{KO}(E, D, Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E, D; \Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, D, E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, D, E) \cap \mathfrak{KO}(E, D, E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, respectively.

We had the following diagram for triples $(E, D; \Gamma)$:

$$\begin{gathered}\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space} \\ \Longrightarrow \text{H-space} \Longrightarrow \text{G-convex space} \Longrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space} \\ \Longrightarrow \text{Partial KKM space} \Longrightarrow \text{Abstract convex space.}\end{gathered}$$

3. General KKM Theorems A, B, and C

In [13,14,16], we gave standard forms of the KKM type theorems as follows.

Theorem A. *Let $(E, D; \Gamma)$ be a partial KKM space [resp., a KKM space], and $G : D \multimap E$ a multimap satisfying*

(1) G has closed [resp., open] values; and

(2) $\Gamma_N \subset G(N)$ for any $N \in \langle D \rangle$ (that is, G is a KKM map).

Then $\{G(y)\}_{y \in D}$ has the finite intersection property.

Further, if

(3) $\bigcap_{y \in M} \overline{G(y)}$ is compact for some $M \in \langle D \rangle$,

then we have

$$\bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Recall that Theorem A is a simple consequence of the definitions of the partial KKM principle or the KKM principle.

Consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y \in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y \in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y \in D} \overline{G(y)} = \overline{\bigcap_{y \in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y \in D} \overline{G(y)} = \bigcap_{y \in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

From the partial KKM principle we have a whole intersection property of the Fan type as follows.

Theorem B. *Let $(E, D; \Gamma)$ be a partial KKM space and $G : D \multimap E$ a map such that*

(1) $\overline{G}$ is a KKM map [that is, $\Gamma_A \subset \overline{G}(A)$ for all $A \in \langle D \rangle$]; and

(2) there exists a nonempty compact subset K of E such that either

(i) $\bigcap\{\overline{G(y)} \mid y \in M\} \subset K$ for some $M \in \langle D \rangle$; or

(ii) *for each* $N \in \langle D \rangle$, *there exists a compact* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$ *and*

$$\overline{L_N} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K.$$

Then we have $K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset$.

Furthermore,

(α) *if* G *is transfer closed-valued, then* $K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$;

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.

Recall that conditions (i) and (ii) in Theorem B are usually called the *compactness conditions* or the *coercivity conditions*, and (ii) has numerous variations or particular forms appeared in a very large number of literature. Note that Theorem B can be easily deduced from the compact case of Theorem A; see [13, 14].

Theorem B can be extended for $F \in \mathfrak{KC}(E, D, Z)$ instead of $1_E \in \mathfrak{KC}(E, D, E))$ as the following in [13,14]:

Theorem C. *Let* $(E, D; \Gamma)$ *be an abstract convex space,* Z *a topological space,* $F \in \mathfrak{KC}(E, D, Z)$, *and* $G : D \multimap Z$ *a map such that*

(1) $\overline{G}$ *is a KKM map w.r.t.* F; *and*

(2) *there exists a nonempty compact subset* K *of* Z *such that either*

(i) $K \supset \bigcap\{\overline{G(y)} \mid y \in M\}$ *for some* $M \in \langle D \rangle$; *or*

(ii) *for each* $N \in \langle D \rangle$, *there exists a* Γ*-convex subset* L_N *of* X *relative to some* $D' \subset D$ *such that* $N \subset D'$, $\overline{F(L_N)}$ *is compact, and*

$$K \supset \overline{F(L_N)} \cap \bigcap\{\overline{G(y)} \mid y \in D'\}.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if* G *is transfer closed-valued, then* $\overline{F(E)} \cap K \cap \bigcap\{G(y) \mid y \in D\} \neq \emptyset$; *and*

(β) *if* G *is intersectionally closed-valued, then* $\bigcap\{G(y) \mid y \in D\} \neq \emptyset$.

In Theorem C, let $\Lambda_A := F(\Gamma_A)$ for each $A \in \langle D \rangle$. Then $(Z, D; \Lambda)$ is called the *abstract convex space induced by* F. In our recent work [17], by replacing $(E, D; \Gamma)$, K, L_N in Theorem B by $(Z, D; \Lambda)$, $\overline{F(E)} \cap K$, $\overline{F(L_N)}$, respectively, we obtained Theorem C. Consequently, we showed that Theorem A(closed case), Theorem B, and Theorem C are mutually equivalent.

In [17], the following was basic.

Proposition. *For an abstract convex space* $(E, D; \Gamma)$, *the corresponding abstract convex space* $(Z, D; \Lambda)$ *induced by* $F : D \multimap Z$ *is a partial KKM space if and only if* $F \in \mathfrak{KC}(E, D, Z)$.

The abstract convex space $(Z, D; \Lambda)$ *induced by* $F : D \multimap Z$ *is a KKM space if and only if* $F \in \mathfrak{KC}(E, D, Z) \cap \mathfrak{KO}(E, D, Z)$.

4. A basic coincidence theorem

From the KKM theorem C, we can deduce the following coincidence theorem of the Fan-Browder type.

Theorem D. *Let* $(E, D; \Gamma)$ *be an abstract convex space,* Z *a topological space,* $F \in \mathfrak{KC}(E, D, Z)$, *and* $S : D \multimap Z$, $T : E \multimap Z$ *maps. Suppose that*

(1) *for each* $z \in F(E)$, *we have* $\mathrm{co}_\Gamma \mathrm{S}^-(\mathrm{z}) \subset \mathrm{T}^-(\mathrm{z})$

(2) *there exists a nonempty compact subset* K *of* Z *such that either*

(i) $\bigcap_{y \in M} \overline{Z \setminus S(y)} \subset K$ *for some* $M \in \langle D \rangle$; *or*

(ii) *for each* $N \in \langle D \rangle$, *there exists a* Γ*-convex subset* L_N *of* E *relative to some* $D' \subset D$ *such that* $N \subset D'$, $\overline{F(L_N)}$ *is compact, and*

$$\overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{Z \setminus S(y)} \subset K.$$

(α) *If* S *is transfer open-valued and* $\overline{F(E)} \cap K \subset S(D)$, *then there exist* $\bar{x} \in E$ *and* $\bar{z} \in \overline{F(E)} \cap K$ *such that* $\bar{z} \in F(\bar{x}) \cap T(\bar{x})$.

(β) *if* S *is unionly open-valued and* $Z = S(D)$, *then there exists an* $\bar{x} \in E$ *such that* $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

Proof of Theorem D using Theorem C. Suppose that $F(x) \cap T(x) = \emptyset$ and hence $F(x) \subset Z \setminus T(x)$ for all $x \in E$. Let

$G(y) := Z \setminus S(y)$ for all $y \in D$; and $H(x) := Z \setminus T(x)$ for all $x \in E$.

Then we have

(3) $F(x) \subset H(x)$ for all $x \in E$.

From (1.1) and (1.3), it follows that

(4) G is a KKM map w.r.t. F.

In fact, suppose that there exists an $N \in \langle D \rangle$ such that $F(\Gamma_N) \not\subset G(N)$. Then there exist $x \in \Gamma_N$ and $z \in F(x)$ such that $z \notin G(y) = Z \setminus S(y)$ for all $y \in N$. Hence $z \in S(y)$ or $y \in S^-(z)$ for all $y \in N$, that is, $N \in \langle S^-(z) \rangle$. Therefore, $\Gamma_N \subset T^-(z)$ by (1.1). Since $x \in \Gamma_N \subset T^-(z)$, we have $z \in T(x)$ and hence $z \notin H(x)$. Since $z \in F(x)$, this contradicts (3). Therefore (4) holds.

Note that (4) and (2) imply the requirements (1) and (2) of Theorem C, resp. Now by Theorem C, there exists $z_0 \in \overline{F(E)} \cap K \cap \bigcap\{\overline{G(y)} \mid y \in D\}$.

Case (α). Since G is transfer closed-valued, $z_0 \in \overline{F(E)} \cap K$ such that $z_0 \in \bigcap\{\overline{G(y)} \mid y \in D\} = \bigcap\{G(y) \mid y \in D\} = \bigcap\{Z \setminus S(y) \mid y \in D\}$ and hence $z_0 \notin S(y)$ for all $y \in D$. This contradicts $\overline{F(E)} \cap K \subset S(D)$.

Case (β). Since G is intersectionally closed-valued, by Theorem C, there exists $z_0 \in \bigcap\{G(y) \mid y \in D\}$, that is, $z_0 \notin S(y)$ for all $y \in D$. This contradicts $Z = S(D)$.

Therefore our proof is complete.

5. Particular forms in our earlier works

In this section, we recollect that several particular forms of Theorems A-D were applied to establish our earlier foundational works on the KKM theory for each of convex spaces, H-spaces, G-convex spaces, and abstract convex spaces.

5.1. FPTA 1992 [1]

Abstract: From a Lefschetz type fixed point theorem for composites of acyclic maps, we obtain a general Fan-Browder type coincidence theorem, which can be shown to be equivalent to a matching theorem and a KKM type theorem. From the main result, we deduce the Himmelberg type fixed point theorem for acyclic compact multifunctions, acyclic versions of general geometric properties of convex sets, abstract variational inequality theorems, new minimax theorems, and non-continuous versions of the Brouwer and Kakutani type fixed point theorems with very generous boundary conditions.

This paper is based on the following particular form of Theorem D.

Theorem 1 ([1]). *Let D be a nonempty subset of a convex space X, Y a Hausdorff space, $S : D \to 2^Y$, $T : X \to 2^Y$ multifunctions, $F : X \multimap Y$ a u.s.c. multifunction with compact acyclic values, and K a nonempty compact subset of Y. Suppose that*

(1.1) *for each $x \in D$, $Sx \subset Tx$ and Sx is compactly open;*

(1.2) *for each $y \in F(X)$, $T^{-}y$ is convex;*

(1.3) $\mathrm{cl}\, F(X) \cap K \subset S(D)$; *and*

(1.4) *for each $N \in \langle D \rangle$, there exists a compact convex subset L_N of X containing N such that $x \in L_N \backslash F^{+}(K)$ implies $Fx \subset S(L_N \cap D)$.*

Then T and F have a coincidence point $x_0 \in X$; that is, $Tx_0 \cap Fx_0 \neq \emptyset$.

Particular forms. Given in earlier works of Park and S. Y. Chang; see [1].

5.2. JKMS 1994 [2]

From Introduction: The purpose in [2] is, first, to establish some coincidence theorems for composites of multifunctions including a class of very general u.s.c. maps. Consequently, we obtain generalizations of main results of some previous works to a class of maps which properly includes that of multifunctions factorizable by Kakutani or acyclic maps. Secondly, we show that fundamental theorems in the KKM theory can be obtained in far-reaching generalized forms related to such class of maps. Those are the KKM theorem, the matching theorem, the Fan-Browder fixed point theorem, the Ky Fan minimax inequality, analytic alternatives, geometric properties of convex sets, and others.

This paper is based on the following form of Theorem D.

Theorem 5 ([2]). *Let (X, D) be a convex space, Y a Hausdorff space, $S : D \to 2^Y$, $T : X \to 2^Y$ multifunctions, and $F \in \mathfrak{A}_c^{\kappa}(X, Y)$. Suppose that*

(5.1) *for each* $x \in D$, $Sx \subset Tx$ *and* Sx *is compactly open;*
(5.2) *for each* $y \in F(X)$, $T^{-}y$ *is* D*-convex;*
(5.3) *there exists a nonempty compact subset* K *of* Y *such that* $\overline{F(X)} \cap K \subset S(D)$*; and*
(5.4) *for each* $N \in \langle D \rangle$, *there exists a compact* D*-convex subset* L_N *of* X *containing* N *such that* $F(L_N) \backslash K \subset S(L_N \cap D)$.
Then F *and* T *have a coincidence point.*

Here $\mathfrak{A}_c^{\kappa}(X, Y)$ is the *admissible class* of multimaps in the sense of Park.

5.3. JKMS 1995 [3]

From Introduction: In [3], we extend the main coincidence theorem of [2] to H-spaces and apply it to obtain a far-reaching generalization of the KKM theorem and a fixed point theorem for H-spaces. Many of the main results in previous papers are extended and unified.

This paper [3] is based on the following particular form of Theorem D.

Theorem 1 ([3]). *Let* $(X, D; \Gamma)$ *be an* H*-space,* Y *a Hausdorff space,* $F \in \mathfrak{A}_c(X, Y)$, *and* K *a nonempty compact subset of* Y. *Let* $S : D \to 2^Y$ *and* $T : X \to 2^Y$ *satisfy the following:*
(1.1) *for each* $x \in D$, Sx *is (compactly) open in* Y*;*
(1.2) *for each inF*(X), $M \in \langle S^{-} \rangle$ *implies* $\Gamma_M \subset T^{-}y$*;*
(1.3) $\overline{F(X)} \cap K \subset S(D)$*; and*
(1.4) *suppose that either*
(i) $Y \backslash K \subset S(M)$ *for some* $M \in \langle D \rangle$*; or*
(ii) *for each* $N \in \langle D \rangle$, *there exists a compact* H*-subspace* L_N *of* X *containing* N *such that* $F(L_N) \backslash K \subset S(L_N \cap D)$.
Then T *and* F *has a coincidence point* $x_0 \in X$*; that is,* $Tx_0 \cap Fx_0 \neq \emptyset$.

5.4. JMAA 1996, 1997 [4,5]

Abstract: [4] We defined admissible classes of maps which are general enough to include composites of maps appearing in nonlinear analysis or algebraic topology, and generalized convex spaces which are generalizations of many general convexity structures. In [4] we obtain a coincidence theorem for admissible maps defined on generalized convex spaces. Our new result is applied to obtain an abstract variational inequality, a KKM type theorem, and fixed point theorems.

[5] Recently, we introduced new concept of a generalized convex space. In [5], from a coincidence theorem, we deduce far-reaching generalizations of the KKM theorem, the matching theorem, a whole intersection property, an analytic alternative, the Ky Fan minimax inequality, geometric or section properties, and others on generalized convex spaces.

These papers are based on the following form of Theorem D.

Theorem 1 ([4,5]). *Let* $(X \supset D; \Gamma)$ *be a* G*-convex space,* Y *a Hausdorff space,* $S : D \multimap Y$, $T : X \multimap Y$ *maps, and* $F \in \mathfrak{A}_c^{\kappa}(X, Y)$. *Suppose that*

(1.1) *for each* $x \in D$, $S(x)$ *is compactly open in* Y;

(1.2) *for each* $y \in F(X)$, $\mathrm{co}S^{-}(y) \subset T^{-}(y)$;

(1.3) *there exists a nonempty compact subset* K *of* Y *such that* $\overline{F(X)} \cap K \subset S(D)$; *and*

(1.4) *either*

(i) $Y \backslash K \subset S(M)$ *for some* $M \in \langle D \rangle$; *or*

(ii) *for each* $N \in \langle D \rangle$, *there exists a compact G-convex subset* L_N *of* X *containing* N *such that* $F(L_N) \backslash K \subset S(L_N \cap D)$.

Then there exists an $\bar{x} \in X$ *such that* $F\bar{x} \cap T\bar{x} \neq \emptyset$.

This theorem contains a very large number of previously known results; see [4].

Remark. 1. If X is a convex space with $\Gamma_A = \mathrm{coA}$, then (i) implies (ii). In fact we can choose $L_N = \mathrm{co}(\mathrm{M} \cup \mathrm{N})$. However, in general, we cannot say (i)$\Rightarrow$(ii) for G-convex spaces.

2. Note that the Hausdorffness of Y is necessary for the partition of unity argument in the proof. If F is single-valued, we do not need to assume the Hausdorffness of Y.

3. Note that (1.2) generalizes the following:

(1.2)$'$ *for each* $x \in D$, $Sx \subset Tx$ *and* $T^{-}y$ *is G-convex for each* $y \in F(X)$,

as in previous works of Park for convex spaces and H-spaces.

4. If F is compact, then by putting $K = \overline{F(X)}$, condition (1.4) holds automatically.

Particular forms for compact admissible maps [4].

1. For convex spaces: Browder, Tarafdar and Husain, Ben-El-Mechaiekh et al., Takahashi, Komiya, Granas and Liu, Lassonde, Park et al.

2. For other particular types of G-convex spaces: Komiya, Bielawski, Horvath, and Park and Kim.

Particular forms for non-compact admissible maps[4].

1. For convex spaces: Park, and for H-spaces: Park and Kim.

2. For $\mathbb{V}$ instead of $\mathfrak{A}_c^{\kappa}$: Browder, Tarafdar, Tarafdar and Husain, Ben-El-Mechaiekh et al., Yannelis and Prabhakar, Lassonde, Ko and Tan, Simons, Takahashi, Komiya, Mehta, Mehta and Tarafdar, Sessa, Jiang, McLinden, Granas and Liu, Park, and Chang.

3. For an H-space: Horvath, Ding and Tan, Ding et al., Tarafdar, Chen, and Park.

In [5], the following particular form of Theorem C was given.

Theorem 3 ([5]). *Let* $(X, D; \Gamma)$ *be a G-convex space,* Y *a Hausdorff space, and* $F \in \mathfrak{A}_c^{\kappa}(X, Y)$. *Let* $G : D \multimap Y$ *be a map such that*

(3.1) *for each* $x \in D$, Gx *is (compactly) closed in* Y;

(3.2) *for any* $N \in \langle D \rangle$, $F(\Gamma_N) \subset G(N)$; *and*

(3.3) *there exist a nonempty compact subset* K *of* Y *such that either*

(i) $\bigcap\{Gx : x \in M\} \subset K$ *for some* $M \in \langle D \rangle$; *or*

(ii) *for each* $N \in \langle D \rangle$, *there exists a compact* G*-convex subset* L_N *of* X *containing* N *such that* $F(L_N) \cap \bigcap\{Gx : x \in L_N \cap D\} \subset K$.

Then $\overline{F(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

This also contains a large number of previous results; see [5].

Particular forms 1. The origin of Theorem 3: Sperner and Knaster-Kuratowski-Mazurkiewicz.

2. For a convex space X: Fan, Lassonde, Chang, and Park. Also Sehgal-Singh-Whitfield, Shioji, Liu, Chang-Zhang, and Guillerme.

3. For an H-space X: Horvath, Bardaro-Ceppitelli, Ding-Kim-Tan, Park, and Ding.

5.5. KJCAM 2000 [7]

Abstract: In [7], we introduce fundamental results in the KKM theory for G-convex spaces which are equivalent to the Brouwer theorem, the Sperner lemma, and the KKM theorem. Those results are all abstract versions of known corresponding ones for convex subsets of topological vector spaces. Some earlier applications of those results are indicated. Finally, we give a new proof of the Himmelberg fixed point theorem and G-convex space versions of the von Neumann type minimax theorem and the Nash equilibrium theorem as typical examples of applications of our theory.

This paper [7] is based on the following KKM theorem for G-convex spaces particular to Theorem A.

Theorem 1 ([7]). *Let* $(X, D; \Gamma)$ *be a* G*-convex space and* $F : D \multimap X$ *a map such that*

(1.1) F *has (compactly) closed [resp., open] values; and*

(1.2) F *is a KKM map.*

Then $\{F(z)\}_{z \in D}$ *has the finite intersection property.*

Further, if F *has (compactly) closed values and if*

(1.3) $\bigcap_{z \in M} F(z)$ *is compact for some* $M \in \langle D \rangle$,

then we have

$$\bigcap_{z \in D} F(z) \neq \emptyset.$$

5.6. JKMS 2008 [8]

Abstract: We introduce a new concept of abstract convex spaces and a multimap class $\mathfrak{K}$ having certain KKM property. From a basic KKM type theorem for a $\mathfrak{K}$-map defined on an abstract convex space without any topology, we deduce ten equivalent formulations of the theorem. As applications of the equivalents, in the frame of abstract convex topological spaces, we obtain Fan-Browder type fixed point theorems, almost fixed point theorems for multimaps, mutual relations between the map classes $\mathfrak{K}$ and $\mathfrak{B}$, variational inequalities, the von Neumann type minimax theorems, and the Nash equilibrium theorems.

This paper [8] is based on the following variant of Theorem A.

Theorem 1 ([8]). *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, and $F : E \multimap Z$ a map. Then $F \in \mathfrak{K}(E, Z)$ if and only if for any map $G : D \multimap Z$ satisfying*

(1.1) *$F(\Gamma_N) \subset G(N)$ for any $N \in \langle D \rangle$,*

we have $F(E) \cap \bigcap\{G(y) \mid y \in N\} \neq \emptyset$ for each $N \in \langle D \rangle$.

Here, a multimap $F : E \multimap Z$ is called *a $\mathfrak{K}$-map* if, for a KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

5.7. JNCA 2008 [9]

Abstract: A KKM space is an abstract convex space satisfying an abstract form of the KKM theorem and its 'open' version. We give several characterizations of KKM spaces as abstract convex spaces satisfying one of the properties of matching, intersection, geometric or section, Fan-Browder type fixed point, or existence of maximal elements. We deduce fundamental results on KKM spaces; for example, several whole intersection properties, analytic alternatives, minimax inequalities, variational inequalities, etc. These results are all abstract versions of known corresponding ones for convex subsets of topological vector spaces, convex spaces due to Lassonde, c-spaces due to Horvath, G-convex spaces due to the author, and their variations. Some earlier applications of those results are indicated. Moreover, it is noted that many of the results are mutually equivalent.

This paper [9] is based on several equivalent formulations of Theorem A. The following is one of them.

Theorem 4.1 ([9]). *An abstract convex space $(X, D; \Gamma)$ satisfies the partial KKM principle iff for any maps $S : D \multimap X$, $T : X \multimap X$ satisfying*

(1) *S has closed values;*

(2) *for each $x \in X$, $\mathrm{co}_\Gamma(D \setminus S^-(x)) \subset X \setminus T^-(x)$; and*

(3) *$x \in T(x)$ for each $x \in X$,*

$\{S(z)\}_{z \in D}$ has the finite intersection property.

An abstract convex space $(X, D; \Gamma)$ is a KKM space iff the above condition also holds for any open-valued map S.

5.8. NA 2010 [11]

Abstract: The partial KKM principle for an abstract convex space is an abstract form of the classical KKM theorem. A KKM space is an abstract convex space satisfying the partial KKM principle and its "open" version. In [11], we clearly derive a sequence of a dozen statements which characterize the KKM spaces and equivalent formulations of the partial KKM principle. As their applications, we add more than a dozen statements including

generalized formulations of von Neumann minimax theorem, von Neumann intersection lemma, the Nash equilibrium theorem, and the Fan type minimax inequalities for any KKM spaces. Consequently, this paper [11] unifies and enlarges previously known several proper examples of such statements for particular types of KKM spaces.

This paper [11] begins with the following form of Theorem A.

(0) The KKM principle. *For any closed-valued [resp., open-valued] KKM map* $G : D \multimap E$, *the family* $\{G(z)\}_{z \in D}$ *has the finite intersection property.*

This paper [11] contains some incorrectly stated statements such as (VI), Theorem 4, (XVI), and (XVII). These can be corrected easily.

5.9. NA 2011 [12]

Abstract: In [12], we obtain a new KKM type theorem for intersectionally closed-valued KKM maps and some useful new basic consequences. Typical examples of them are abstract forms of Fan's matching theorem, Fan's geometric lemma, the Fan-Browder fixed point theorem, maximal element theorems, Fan's minimax inequality, variational inequalities, and others.

The paper [12] is based on Theorem B.

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